

# Quantum bosonic string energy-momentum tensor in Minkowski space-time.

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## Abstract

The quantum energy-momentum tensor  $\hat{T}^{\mu\nu}(x)$  is computed for strings in Minkowski space-time. We compute its expectation value for different physical string states both for open and closed bosonic strings. The states considered are described by normalizable wave-packets in the center of mass coordinates. We find in particular that  $\hat{T}^{\mu\nu}(x)$  is **finite** which could imply that the classical divergence that occurs in string theory as we approach the string position is removed at the quantum level as the string position is smeared out by quantum fluctuations.

For massive string states the expectation value of  $\hat{T}^{\mu\nu}(x)$  vanishes at leading order (genus zero). For massless string states it has a non-vanishing value which we explicitly compute and analyze for both spherically and cylindrically symmetric wave packets. The energy-momentum tensor components propagate outwards as a massless lump peaked at  $r = t$ .

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# 1 Introduction.

String theory has emerged as the most promising candidate to reconcile general relativity with quantum mechanics and unify gravity with the other fundamental interactions. It makes sense therefore to investigate the gravitational consequences of strings as we approach the Planck scale.

When particles scatter at energies of the order of or larger than the Planck mass, the interaction that dominates their collision is the gravitational one, at these energies the picture of particle fields or strings in flat space-time ceases to be valid, the curved space-time geometry created by the particles has to be taken into account. This has been our motivation to investigate the possible gravitational effects arising from an isolated quantum bosonic string living in a flat space-time background, so we may begin a study of the scattering process of strings merely by the gravitational interaction between them. The systematic study of quantum strings in physically relevant curved space-times was started in [1] and is reviewed *in extenso* in ref.[2].

In this paper, we calculate the energy-momentum tensor of both closed and open quantum bosonic strings in 3+1 dimensions. Our target space is the direct product of four dimensional Minkowski space-time times a compact manifold taking care of conformal anomalies.

It has been shown [3] that the back-reaction for a classical bosonic string in 3 + 1 dimensions has a logarithmic divergence when the spacetime coordinate  $x \rightarrow X(\sigma)$  that is, when we approach the core of the string. This divergence is absorbed into a renormalization of the string tension. Copeland et al [3] showed that by demanding that both it and the divergence in the energy-momentum tensor vanish forces the string to have the couplings of compactified  $N = 1$ ,  $D = 10$  supergravity. In this paper we are able to see that when we take into account the quantum nature of the strings we lose all information regarding the position of the string and therefore any divergences that may appear when one calculates the back-reaction of quantum strings are not related to our position with respect to the position of the string.

In the present work, the energy-momentum tensor of the string,  $\hat{T}^{\mu\nu}$ , is a quantum operator and it may be regarded as the vertex operator for the emission (absorption) of gravitons in the case of closed strings or, for the open string case, as the vertex operator for the emission (absorption) of massive spin 2 particles. We compute its expectation value in one-particle string states, choosing for the string center of mass wave function a wave packet centered at the origin.

Our results dramatically depend on the mass of the string state chosen. For massive states the expectation value of the string energy-momentum tensor identically vanishes at leading order (genus zero), thus massive states of closed and open strings give no contribution to  $\hat{T}^{\mu\nu}$ .

Massless string states (such as gravitons, photons or dilatons) yield a non-zero results which turn out to be spin independent, that is the same expression for gravitons, photons and dilatons.

We consider spherically symmetric and cylindrically symmetric configurations. The components for the string energy density and energy flux behave like massless waves, with the string energy being radiated outwards as a massless lump peaked at  $r = t$  (for the spherically symmetric case). We provide integral representations for  $\langle \hat{T}^{\mu\nu}(r, t) \rangle$  [eq.(14)]. After exhibiting the tensor structure of  $\langle \hat{T}^{\mu\nu} \rangle$  [eqs.(16) and (19)], the asymptotic behaviour of

$\langle \hat{T}^{\mu\nu}(r, t) \rangle$  for  $r \rightarrow \infty$  and  $t$  fixed and for  $t \rightarrow \infty$  with  $r$  fixed is computed. In the first regime the energy density and the stress tensor decay as  $r^{-1}$  whereas the energy flux decays as  $r^{-2}$ . For  $t \rightarrow \infty$  with  $r$  fixed, the energy density tends to  $0^-$  as  $t^{-6}$ . That is, the spherical wave leaves behind a rapidly vanishing negative energy density.

For cylindrically symmetric configurations,  $\langle \hat{T}^{\mu\nu}(\rho, t) \rangle$  propagates as outgoing (plus ingoing) cylindrical waves. For large  $\rho$  and fixed  $t$  the energy density decays as  $1/\rho$  and the energy flux decays as  $1/\rho^2$ . For large  $t \rightarrow \infty$  with  $\rho$  fixed, the same phenomenon of a negative energy density vanishing as  $t^{-6}$  appears.

We restrict ourselves to bosonic strings in this paper. We expect analogous results from superstrings, since only massless string states contribute to the expectation values of  $\hat{T}^{\mu\nu}$ .

The structure of this paper is as follows: in section 2, we consider the string energy-momentum tensor as a quantum operator and discuss its quantum ordering problems, observing that it can be regarded as a vertex operator. We calculate the expectation value of the energy-momentum tensor for different physical states of the strings (for both open and closed strings). In section 3, we calculate the string energy-momentum tensor expectation value for massless states in spherically symmetric configurations, whilst in section 4 we calculate it for cylindrically symmetric configurations which are relevant when we study for example cosmic strings which are essentially very long strings. Finally, in section 5 some final remarks about our results are stressed.

## 2 The string energy-momentum tensor.

The energy-momentum tensor for a classical bosonic string with tension  $(\alpha')^{-1}$  is given by

$$T^{\mu\nu}(x) = \frac{1}{2\pi\alpha'} \int d\sigma d\tau (\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu) \delta(x - X(\sigma, \tau)) \quad (1)$$

and the string coordinates are given in Minkowski space-time by

$$X^\mu(\sigma, \tau) = q^\mu + 2\alpha' p^\mu \tau + i\sqrt{\alpha'} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^\mu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}] \quad (2)$$

for closed strings and

$$X^\mu(\sigma, \tau) = q^\mu + 2\alpha' p^\mu \tau + i\sqrt{\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma \quad (3)$$

for open strings. For closed strings we can set  $\alpha' = 1/2$ , so inserting eq. (2) in eq. (1) and

rewriting the four dimensional delta function in integral form we obtain for closed strings,

$$\begin{aligned} T^{\mu\nu}(x) = & \frac{1}{\pi} \int d\sigma d\tau \frac{d^4\lambda}{(2\pi)^4} \{ p^\mu p^\nu + \frac{p^\mu}{\sqrt{2}} \sum_{n \neq 0} [\alpha_n^\nu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\nu e^{-in(\tau+\sigma)}] + \\ & + \sum_{n \neq 0} (\alpha_n^\mu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}) \frac{p^\nu}{\sqrt{2}} + \\ & + \sum_{n \neq 0} \sum_{m \neq 0} [\alpha_n^\mu \tilde{\alpha}_m^\nu e^{-in(\tau-\sigma)} e^{-im(\tau+\sigma)} + \tilde{\alpha}_n^\mu \alpha_m^\nu e^{-in(\tau+\sigma)} e^{-im(\tau-\sigma)}] \} e^{i\lambda \cdot x} e^{-i\lambda \cdot X(\sigma, \tau)}. \end{aligned} \quad (4)$$

We can write

$$X(\sigma, \tau) = X_{cm} + X_+ + X_-,$$

where  $X_{cm} = q + p\tau$  is the centre of mass coordinate and  $X_+$  and  $X_-$  refer to the terms with  $\alpha_{n>0}$  and  $\alpha_{n<0}$  in  $X(\sigma, \tau)$  respectively. In this way, we can see now that our energy-momentum tensor has the same form as that of a vertex operator, when we recall that the vertex operator for closed strings has to have conformal dimension 2 whereas for open strings it has to have conformal dimension 1 [10] in order that the string is anomaly free.

Eq.(1) is meaningful at the classical level. However, at the quantum level, one must be careful with the order of the operators since  $\dot{X}^\mu$  and  $\dot{X}^\nu$  do not commute with  $X(\sigma, \tau)$ . We shall define the quantum operator  $\hat{T}^{\mu\nu}(x)$  by symmetric ordering. That is,

$$\begin{aligned} \hat{T}^{\mu\nu}(x) \equiv & \frac{1}{2\pi\alpha'} \int d\sigma d\tau \\ & \left\{ \frac{1}{3} \left[ \dot{X}^\mu \dot{X}^\nu \delta(x - X(\sigma, \tau)) + \dot{X}^\mu \delta(x - X(\sigma, \tau)) \dot{X}^\nu + \delta(x - X(\sigma, \tau)) \dot{X}^\mu \dot{X}^\nu \right] \right. \\ & \left. - X'^\mu X'^\nu \delta(x - X(\sigma, \tau)) \right\} \end{aligned} \quad (5)$$

This definition ensures hermiticity:

$$\hat{T}^{\mu\nu}(x)^\dagger = \hat{T}^{\mu\nu}(x)$$

Let us consider a string on a mass and spin eigenstate with a center of mass wave function  $\varphi(\vec{p})\delta(p^0 - \sqrt{\vec{p}^2 + m^2})$ . An on-shell scalar string state is then

$$|\Psi\rangle = \int d^4p \varphi(\vec{p}) \delta(p^0 - \sqrt{\vec{p}^2 + m^2}) |\vec{p}\rangle.$$

Here we assume the extra space-time dimensions (beyond four) to be appropriately compactified, and consider string states in the physical (uncompactified) four dimensional Minkowski space-time.

Now, normal ordering eq.(4) using eq.(2) and taking the expectation value with respect to the fundamental scalar state (tachyonic), we get

$$\begin{aligned} \frac{\langle \Psi | \hat{T}^{\mu\nu}(x) | \Psi \rangle}{\langle \Psi | \Psi \rangle} \equiv \langle \hat{T}^{\mu\nu}(x) \rangle &= \frac{1}{3\pi} \int \frac{d^4\lambda}{(2\pi)^4} d^4p_1 d^4p_2 d\sigma d\tau e^{i\lambda \cdot x} [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + p_1^\mu p_2^\nu] \\ &\langle p_1 | e^{-i\lambda \cdot X_{cm}} | p_2 \rangle \varphi^*(\vec{p}_1) \varphi(\vec{p}_2) \delta(p_1^0 - \sqrt{\vec{p}_1^2 + m_1^2}) \delta(p_2^0 - \sqrt{\vec{p}_2^2 + m_2^2}). \end{aligned} \quad (6)$$

Writing

$$\langle p_1 | e^{-i\lambda \cdot X_{cm}} | p_2 \rangle = e^{-i\frac{\tau\lambda^2}{2} - i\lambda p_2 \tau} \delta^4(\lambda + p_1 - p_2),$$

eq.(6) becomes

$$\begin{aligned} \langle \hat{T}^{\mu\nu}(x) \rangle &= \frac{2}{3} \int \frac{d^4\lambda}{(2\pi)^4} d^4p_1 d^4p_2 d\tau e^{i\lambda \cdot x} [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + p_1^\mu p_2^\nu] \\ &e^{-i\frac{\tau\lambda^2}{2} - i\lambda p_2 \tau} \delta^4(\lambda + p_1 - p_2) \varphi^*(\vec{p}_1) \varphi(\vec{p}_2) \delta(p_1^0 - \sqrt{\vec{p}_1^2 + m_1^2}) \delta(p_2^0 - \sqrt{\vec{p}_2^2 + m_2^2}). \end{aligned}$$

Performing the  $\lambda$  and  $\tau$  integrals

$$\begin{aligned} \langle \hat{T}^{\mu\nu}(x) \rangle &= \frac{4}{3(2\pi)^3} \int d^4 p_1 d^4 p_2 e^{i(p_2 - p_1) \cdot x} [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + p_1^\mu p_2^\nu] \\ &\delta((3p_2 - p_1) \cdot (p_2 - p_1)) \varphi^*(\vec{p}_1) \varphi(\vec{p}_2) \delta(p_1^0 - \sqrt{\vec{p}_1^2 + m_1^2}) \delta(p_2^0 - \sqrt{\vec{p}_2^2 + m_2^2}). \end{aligned} \quad (7)$$

The calculation for open strings is obtained by substituting eq.(3) into eq.(5), with the result (setting  $\alpha' = 1$ )

$$\begin{aligned} \langle \hat{T}^{\mu\nu}(x) \rangle &= \frac{2}{3(2\pi)^3} \int d^4 p_1 d^4 p_2 e^{i(p_2 - p_1) \cdot x} [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + p_1^\mu p_2^\nu] \\ &\delta((3p_2 - p_1) \cdot (p_2 - p_1)) \varphi^*(\vec{p}_1) \varphi(\vec{p}_2) \delta(p_1^0 - \sqrt{\vec{p}_1^2 + m_1^2}) \delta(p_2^0 - \sqrt{\vec{p}_2^2 + m_2^2}). \end{aligned}$$

The extra factor two in eq.(7) comes from the fact that, for closed strings, we have two independent sets of oscillation modes.

Now, in general, we want to take the expectation value with respect to particle states with higher (mass)<sup>2</sup> and spin than the tachyonic case. Since we are interested in expectation values we must have the same particle in both states, hence  $m_1 = m_2 = m$ . As we shall show later, only for massless particle states has the energy-momentum tensor a non-zero expectation value. Therefore, let us first consider massless string states.

For the closed string there is the graviton

$$|\vec{p}; s\rangle = P_s^{il}(n) \tilde{\alpha}_{-1}^l \alpha_{-1}^i |\vec{p}\rangle \quad (8)$$

and the dilaton

$$|\vec{p}\rangle = P^{il}(n) \tilde{\alpha}_{-1}^l \alpha_{-1}^i |\vec{p}\rangle \quad (9)$$

where  $\vec{p}$  is the momentum ( $p^2 = 0$ ),  $s = \pm$  labels the graviton helicity,  $P_s^{il}(n)$ ,  $1 \leq i, l \leq 3$  projects into the spin 2 graviton states and  $P^{il}(n)$  into the (scalar) dilaton state,

$$\begin{aligned} n^i &\equiv \frac{p^i}{|\vec{p}|} \\ P_s^{il}(n) &= P_s^{li}(n) \quad , \quad n^i P_s^{il}(n) = 0 \\ P_s^{ll}(n) &= 0 \quad , \quad P_s^{il}(n) P_{s'}^{il}(n) = \delta_{ss'} \\ P_s^{il}(n) &= P_s^{il}(-n) \quad , \quad P^{il}(n) \equiv \delta^{il} - n^i n^l. \end{aligned}$$

The massless vector states (photons) for open strings are given by

$$|\vec{p}; i\rangle = P^{il}(n) \alpha_{-1}^l |\vec{p}\rangle.$$

In analogy to the tachyon case eq.(6) we obtain for closed strings

$$\begin{aligned} \langle \vec{p}_1 | \tilde{\alpha}_1^l \alpha_1^i \hat{T}^{\mu\nu}(x) \tilde{\alpha}_{-1}^j \alpha_{-1}^m | \vec{p}_2 \rangle &= \frac{1}{12\pi} \int \frac{d^4 \lambda}{(2\pi)^4} d^4 p_1 d^4 p_2 d\sigma d\tau A^{ljim} [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + p_1^\mu p_2^\nu] \\ &\langle p_1 | e^{i\lambda \cdot X} e^{-i\lambda \cdot X_{cm}} | p_2 \rangle \varphi^*(\vec{p}_1) \varphi(\vec{p}_2) \delta(p_1^0 - \sqrt{\vec{p}_1^2 + m^2}) \delta(p_2^0 - \sqrt{\vec{p}_2^2 + m^2}), \end{aligned} \quad (10)$$

where

$$A^{ljim} = 4\delta^{lj}\delta^{im} - 2\delta^{lj}\lambda^i\lambda^m - 2\delta^{im}\lambda^l\lambda^j + \lambda^l\lambda^j\lambda^i\lambda^m.$$

Whereas for the open string case we get,

$$\begin{aligned} \langle \vec{p}_1 | \alpha_1^i \hat{T}^{\mu\nu}(x) \alpha_{-1}^m | \vec{p}_2 \rangle &= \frac{2}{3} \int d^4 p_1 d^4 p_2 \frac{e^{i(p_2 - p_1) \cdot X}}{(2\pi)^3} [\delta^{im} (p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + p_1^\mu p_2^\nu) \\ &\quad - \frac{3}{2} \eta^{\mu\nu} (p_2 - p_1)^i (p_2 - p_1)^m] \delta((3p_2 - p_1) \cdot (p_2 - p_1)) \\ &\quad \varphi^*(\vec{p}_1) \varphi(\vec{p}_2) \delta(p_1^0 - \sqrt{\vec{p}_1^2 + m^2}) \delta(p_2^0 - \sqrt{\vec{p}_2^2 + m^2}). \end{aligned} \quad (11)$$

having performed the  $\sigma$ ,  $\tau$  and  $\lambda$  integrations.

Projecting on the massless physical states (8) and (9) and integrating over  $p_1^0$  and  $p_2^0$ , eqs.(10) and (11) become

$$\begin{aligned} \langle p_1, s | \hat{T}^{\mu\nu}(x) | p_2, s \rangle &= \frac{1}{6(2\pi)^3} \int d^3 p_1 d^3 p_2 e^{i(p_2 - p_1) \cdot x} (p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + p_1^\mu p_2^\nu) \\ &\quad \varphi^*(\vec{p}_1) \varphi(\vec{p}_2) \delta(p_2 \cdot p_1), \end{aligned} \quad (12)$$

$$\begin{aligned} \langle p_1 | \hat{T}^{\mu\nu}(x) | p_2 \rangle &= \frac{1}{6(2\pi)^3} \int d^3 p_1 d^3 p_2 e^{i(p_2 - p_1) \cdot x} (p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + p_1^\mu p_2^\nu) \\ &\quad \varphi^*(\vec{p}_1) \varphi(\vec{p}_2) \delta(p_2 \cdot p_1) \end{aligned} \quad (13)$$

respectively. It is easy to check that this energy-momentum tensor is indeed conserved.

We will evaluate eq.(12) and eq.(13) shortly. For the massive case ( $m \neq 0$ ) we will now show that the equivalent result has to be identically zero. It is convenient to parametrize the momenta  $p_1$  and  $p_2$  for  $m \neq 0$  in eq.(10) and eq.(11) as follows:

$$p_1 = m(\cosh u, \sinh u \hat{u}_1) \quad , \quad p_2 = m(\cosh v, \sinh v \hat{u}_2)$$

with  $\hat{u}_1$  and  $\hat{u}_2$  unit three-dimensional vectors and  $u, v \geq 0$ . Then,

$$p_2 \cdot p_1 - m^2 = \frac{m^2}{2} [(1 - \hat{u}_1 \cdot \hat{u}_2) \cosh(u + v) + (1 + \hat{u}_1 \cdot \hat{u}_2) \cosh(u - v) - 2]$$

The only real root for  $m \neq 0$  corresponds to

$$\hat{u}_1 = \hat{u}_2 \quad , \quad u - v = 0$$

and arbitrary  $u + v$ . This means that the delta function in eq.(13) **sets**  $p_1 = p_2$  and we arrive to a constant ( $x$ -independent) result for  $\langle \hat{T}^{\mu\nu}(x) \rangle$ .

More precisely, in spherical coordinates

$$\hat{u}_1 = (\cos \alpha \sin \gamma, \sin \alpha \sin \gamma, \cos \gamma) \quad , \quad \hat{u}_2 = (\cos \beta \sin \delta, \sin \beta \sin \delta, \cos \delta)$$

and we find

$$\delta(p_2 \cdot p_1 - m^2) = \frac{8\sqrt{2}\pi}{\sin \gamma (\cosh t - 1) m^3} \sqrt{|p_2 \cdot p_1 - m^2|} \delta(\alpha - \beta) \delta(\gamma - \delta) \delta(u - v)$$

So, actually the expectation value  $\langle \hat{T}^{\mu\nu}(x) \rangle$  vanishes for  $m^2 \neq 0$  since the argument of the  $\sqrt{\cdot}$  vanishes at  $\alpha = \beta, \gamma = \delta, u = v$ .

Actually, the only reasonable constant ( $x$ -independent) value for  $\langle \hat{T}^{\mu\nu}(x) \rangle$  describing a localized object is precisely zero. From this result we can see that open as well as the closed string massive states do not contribute to the expectation value of  $\hat{T}^{\mu\nu}(x)$ .

It seems that any particle emitted by a freely moving quantum string is massless such as photons, gravitons and dilatons.

### 3 $\hat{T}^{\mu\nu}(x)$ for massless string states in spherically symmetric configurations.

Let us now consider the expectation value of the energy-momentum tensor for the massless closed string state given by eq.(12) and eq.(13). Notice that such expectation value is independent of the value of the particle spin (zero, one or two).

For such a case, it is convenient to use the parametrization:

$$p_1 = E_1(1, \hat{u}_1) \quad , \quad p_2 = E_2(1, \hat{u}_2)$$

with  $E_1, E_2 \geq 0$ .

Then we see that  $p_2 \cdot p_1 = 0$  implies  $\hat{u}_1 \cdot \hat{u}_2 = 1$  (unless  $E_1$  or  $E_2$  vanishes). Therefore  $E_1$  **does not need** to be equal to  $E_2$  and we find here a non-constant result.

More precisely, we find in spherical coordinates

$$\delta(p_2 \cdot p_1) = \frac{2\pi}{E_1 E_2 \sin \gamma} \delta(\alpha - \beta) \delta(\gamma - \delta)$$

Inserting this result in eq.(12) and integrating over  $\hat{u}_2$  yields

$$\begin{aligned} \langle \hat{T}^{\mu\nu}(x) \rangle &= \frac{\pi}{3(2\pi)^3} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 \varphi^*(E_1, \hat{u}_1) \varphi(E_2, \hat{u}_1) e^{i(E_2 - E_1)(t - \vec{x} \cdot \hat{u}_1)} \\ &\quad [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + p_1^\mu p_2^\nu] d\hat{u}_1 \end{aligned} \quad (14)$$

where  $d\hat{u}_1 \equiv \sin \gamma d\gamma d\alpha$ ,  $p_1 = E_1(1, \hat{u}_1)$  and now  $p_2 = E_2(1, \hat{u}_1)$ .

Let us consider for simplicity spherically symmetric wave packets  $\varphi(E, \hat{u}) = \varphi(E)$ . If we take  $\vec{x} = (0, 0, r)$  we can then integrate over the angles in eq.(14) with the result

$$\begin{aligned} \langle \hat{T}^{00}(t, r) \rangle &= \frac{1}{6\pi} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 \varphi^*(E_1) \varphi(E_2) e^{i(E_2 - E_1)t} \\ &\quad \frac{\sin(E_2 - E_1)r}{(E_2 - E_1)r} [E_1^2 + E_2^2 + E_1 E_2] \end{aligned} \quad (15)$$

We can relate the result for arbitrary  $x = (t, \vec{x})$  with the special case  $x = (t, 0, 0, z)$  using rotational invariance as follows,

$$\begin{aligned} \langle \hat{T}^{0i}(x) \rangle &= \hat{x}^i C(t, r) \quad , \quad i = 1, 2, 3, \\ \langle \hat{T}^{ij}(x) \rangle &= \delta^{ij} A(t, r) + \hat{x}^i \hat{x}^j B(t, r) \quad , \quad i, j = 1, 2, 3. \end{aligned} \quad (16)$$

Here

$$C(t, r) = \langle \hat{T}^{03}(t, r = z) \rangle$$

$$A(t, r) = \langle \hat{T}^{22}(t, r = z) \rangle, \quad B(t, r) = \langle \hat{T}^{33}(t, r = z) \rangle - \langle \hat{T}^{11}(t, r = z) \rangle \quad (17)$$

with  $\hat{x}^i = \frac{x^i}{r}$  the unit vector, and

$$\begin{aligned} \langle \hat{T}^{11}(t, r = z) \rangle &= -\frac{1}{6\pi} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 \varphi^*(E_1) \varphi(E_2) \frac{e^{i(E_2 - E_1)t}}{(E_2 - E_1)^2 r^2} \\ &\quad \left[ \cos(E_2 - E_1)r - \frac{\sin(E_2 - E_1)r}{(E_2 - E_1)r} \right] [E_1^2 + E_2^2 + E_1 E_2], \\ \langle \hat{T}^{33}(t, r = z) \rangle &= \frac{1}{6\pi} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 [E_1^2 + E_2^2 + E_1 E_2] \\ &\quad \varphi^*(E_1) \varphi(E_2) \frac{e^{i(E_2 - E_1)t}}{(E_2 - E_1)r} \left[ \sin(E_2 - E_1)r + 2 \frac{\cos(E_2 - E_1)r}{(E_2 - E_1)r} \right. \\ &\quad \left. - 2 \frac{\sin(E_2 - E_1)r}{(E_2 - E_1)^2 r^2} \right], \\ \langle \hat{T}^{03}(t, r = z) \rangle &= \frac{i}{6\pi} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 [E_1^2 + E_2^2 + E_1 E_2] \\ &\quad \varphi^*(E_1) \varphi(E_2) \frac{e^{i(E_2 - E_1)t}}{(E_2 - E_1)r} \left[ \cos(E_2 - E_1)r - \frac{\sin(E_2 - E_1)r}{(E_2 - E_1)r} \right]. \quad (18) \end{aligned}$$

The other components satisfy  $\langle \hat{T}^{22}(t, r = z) \rangle = \langle \hat{T}^{11}(t, r = z) \rangle$ ,  $\langle \hat{T}^{01}(t, r = z) \rangle = \langle \hat{T}^{02}(t, r = z) \rangle = \langle \hat{T}^{12}(t, r = z) \rangle = \langle \hat{T}^{13}(t, r = z) \rangle = \langle \hat{T}^{23}(t, r = z) \rangle = 0$ , as they must be from rotational invariance.

As we can see the trace of the expectation value of the string energy-momentum tensor vanishes. In other words the trace of the energy-momentum tensor induced by a quantum string vanishes. Only massless particles are responsible for the field created by the string.

Notice that

$$3A(t, r) + B(t, r) = \langle \hat{T}^{00}(t, r) \rangle$$

due to the tracelessness of the energy-momentum tensor.

The  $r$  and  $t$  dependence in the invariant functions  $\langle \hat{T}^{00}(t, r) \rangle$ ,  $A(t, r)$ ,  $B(t, r)$  and  $C(t, r)$  writing eqs.(15), (17) and (18) as

$$\begin{aligned} \langle \hat{T}^{00}(t, r) \rangle &= \frac{1}{r} [F(t + r) - F(t - r)], \\ A(t, r) &= -\frac{1}{r^2} [H(t + r) + H(t - r)] + \frac{1}{r^3} [E(t + r) - E(t - r)], \\ B(t, r) &= \frac{1}{r} [F(t + r) - F(t - r)] + \frac{3}{r^2} [H(t + r) + H(t - r)] - \frac{3}{r^3} [E(t + r) - E(t - r)], \\ C(t, r) &= -\frac{1}{r} [F(t + r) + F(t - r)] - \frac{1}{r^2} [H(t + r) - H(t - r)]. \quad (19) \end{aligned}$$

where



$$F(x) = \frac{1}{12\pi} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 \varphi(E_1) \varphi(E_2) \frac{\sin(E_2 - E_1)x}{(E_2 - E_1)} \left[ E_1^2 + E_2^2 + E_1 E_2 \right]. \quad (20)$$

Notice that  $F(x) = -H'(x)$  and  $H(x) = E'(x)$ . These relations guarantee the conservation of  $\langle \hat{T}^{\mu\nu}(x) \rangle$ ,

$$\frac{\partial}{\partial t} \langle \hat{T}^{0\nu}(x) \rangle + \frac{\partial}{\partial x^i} \langle \hat{T}^{i\nu}(x) \rangle = 0.$$

We choose a real wave-packet  $\varphi(E)$  decreasing fast with  $E$  and typically peaked at  $E = 0$ . For example a gaussian wave-packet  $\varphi(E)$ :

$$\varphi(E) = \left( \frac{2\alpha}{\pi} \right)^{3/4} e^{-\alpha E^2}, \quad (21)$$

We see from eq.(19) that the energy density  $\langle \hat{T}^{00}(r, t) \rangle$  and the energy flux  $\langle \hat{T}^{0i}(r, t) \rangle$  behave like spherical waves describing the way a string massless state spreads out starting from the initial wave-packet we choose.

In order to compute the asymptotic behaviour of the function  $F(x)$  we change in eq.(20) the integration variables

$$E_2 - E_1 = v\tau/x, \quad E_2 + E_1 = v.$$

We find

$$F(x) = \frac{1}{192\pi} \int_0^\infty v^4 dv \int_0^x \frac{d\tau}{\tau} \varphi\left(\frac{v}{2}\left[1 + \frac{\tau}{x}\right]\right) \varphi\left(\frac{v}{2}\left[1 - \frac{\tau}{x}\right]\right) \sin(v\tau) \left[ 3 - 2\frac{\tau^2}{x^2} - \frac{\tau^4}{x^4} \right].$$

Now, we can let  $x \rightarrow \infty$  with the result

$$F(x) \xrightarrow{x \rightarrow \pm\infty} \pm \frac{1}{4} \int_0^\infty E^4 dE \varphi(E)^2 + \frac{2\varphi(0)^2}{15\pi x^5} + O\left(\frac{1}{x^7}\right)$$

We find through similar calculations,

$$H(x) \xrightarrow{x \rightarrow \pm\infty} -\frac{M}{4} |x| + \frac{N}{4\pi} + \frac{\varphi(0)^2}{30\pi x^4} + O\left(\frac{1}{x^6}\right),$$

$$E(x) \xrightarrow{x \rightarrow \pm\infty} -\frac{M}{8} x^2 \text{sign}(x) + \frac{N x}{4\pi} - \frac{\varphi(0)^2}{90\pi x^3} + O\left(\frac{1}{x^5}\right).$$

Here,

$$M \equiv \int_0^\infty E^4 dE \varphi(E)^2 \quad \text{and} \quad N \equiv \int_0^\infty E^3 dE \varphi(E)^2.$$

For the gaussian wave packet (21) they take the values

$$M = \frac{3}{16\pi\alpha} \quad \text{and} \quad N = \frac{1}{(2\pi)^{3/2} \sqrt{\alpha}}.$$

To gain an insight into the behaviour of  $\langle \hat{T}^{\mu\nu} \rangle$ , we consider the limiting cases:  $r \rightarrow \infty$ ,  $t$  fixed,  $r = t \rightarrow \infty$  and  $t \rightarrow \infty$ ,  $r$  fixed with the following results:

a)  $r \rightarrow \infty$ ,  $t$  fixed

$$\begin{aligned}\langle \hat{T}^{00}(r, t) \rangle &\stackrel{r \rightarrow \infty}{=} \frac{M}{2r} + \frac{4\varphi(0)^2}{15\pi r^6} + O\left(\frac{1}{r^7}\right) , \\ A(r, t) &\stackrel{r \rightarrow \infty}{=} \frac{M}{4r} \left(1 - \frac{t^2}{r^2}\right) - \frac{4\varphi(0)^2}{45\pi r^6} + O\left(\frac{1}{r^7}\right) , \\ B(r, t) &\stackrel{r \rightarrow \infty}{=} -\frac{M}{4r} \left(1 - \frac{3t^2}{r^2}\right) + \frac{8\varphi(0)^2}{15\pi r^6} + O\left(\frac{1}{r^7}\right)\end{aligned}$$

and

$$C(r, t) \stackrel{r \rightarrow \infty}{=} \frac{M t}{2 r^2} + O\left(\frac{1}{r^7}\right) .$$

b)  $r = t$  and large

$$\begin{aligned}\langle \hat{T}^{00}(r, t) \rangle &\stackrel{r=t \rightarrow \infty}{=} \frac{M}{4r} + \frac{\varphi(0)^2}{240\pi r^6} + O\left(\frac{1}{r^7}\right) \\ A(r, t) &\stackrel{r=t \rightarrow \infty}{=} -\frac{5\varphi(0)^2}{1440\pi r^6} + O\left(\frac{1}{r^7}\right) \\ B(r, t) &\stackrel{r=t \rightarrow \infty}{=} \frac{M}{4r} + \frac{7\varphi(0)^2}{480\pi r^6} + O\left(\frac{1}{r^7}\right) \\ C(r, t) &\stackrel{r=t \rightarrow \infty}{=} \frac{3M}{4r} + \frac{N}{4\pi r^2} - \frac{\varphi(0)^2}{480\pi r^6} + O\left(\frac{1}{r^7}\right)\end{aligned}$$

c)  $t \rightarrow \infty$ ,  $r$  fixed

$$\begin{aligned}\langle \hat{T}^{00}(r, t) \rangle &\stackrel{t \rightarrow \infty}{=} -\frac{4\varphi(0)^2}{3\pi} \frac{1}{t^6} + O(t^{-8}) \\ A(r, t) &\stackrel{t \rightarrow \infty}{=} 0 + O(t^{-6}) , \\ B(r, t) &\stackrel{t \rightarrow \infty}{=} 0 + O(t^{-6}) , \\ C(r, t) &\stackrel{t \rightarrow \infty}{=} 0 + O(t^{-7}) .\end{aligned}$$

Thus, as we mentioned earlier the energy density, the energy flux and the components of the stress tensor propagate as spherical outgoing waves. For  $t$  fixed, the energy density decays as  $r^{-1}$  while the energy flux decays as  $r^{-2}$ . This corresponds to a  $r$ -independent radiated energy for large  $r$ .

For  $r$  fixed and large  $t$ , the energy density decays rapidly as  $O(1/t^6)$ . We curiously find a negative energy density in this regime. The spherical wave seems to leave behind a small but negative energy density. Notice that  $T^{00}$  is **not** a positive definite quantity for strings [see eq.(1)].

It must be stressed that the expectation value of the energy-momentum tensor is **everywhere** finite. Even at the potentially troublesome limit, for  $r$  and  $t$  approaching zero, a careful examination of eqs. (15-18) shows that **all**  $\langle \hat{T}^{\mu\nu}(x) \rangle$  components are finite constants in such a limit.

These last results were derived using a gaussian shape for the wave packet. It is clear that the results will be qualitatively similar for any fast-decaying wave function  $\varphi(E)$ .

It should be noted that analogous but not identical results follow for massless waves. A spherically symmetric solution of the wave equation in  $D = 3 + 1$  dimensions

$$\partial^2 \phi(r, t) = 0$$

takes the form,

$$\phi(r, t) = \frac{1}{r} [f(t - r) + g(t + r)] \quad (22)$$

where  $f(x)$  and  $g(x)$  are arbitrary functions. The energy-momentum tensor for such a massless scalar field can be written as:

$$\hat{T}^{\mu\nu}(x)_\phi = \partial^\mu \phi \partial^\nu \phi - \frac{\eta^{\mu\nu}}{2} (\partial\phi)^2. \quad (23)$$

Inserting eq.(22) into eq.(23) yields,

$$\begin{aligned} \hat{T}^{00}(r, t)_\phi &= \frac{1}{r^2} \left\{ f'^2 + g'^2 + \frac{f+g}{2r} \left[ 2(f' - g') + \frac{f+g}{r} \right] \right\}. \\ \hat{T}^{0i}(r, t)_\phi &= \frac{x^i}{r^3} \left[ g'^2 - f'^2 - \frac{1}{r} (f+g)(f' - g') \right] \\ \hat{T}^{ij}(r, t)_\phi &= \frac{1}{2r^2} \left[ \delta^{ij} + \frac{2x^i x^j}{r^2} \right] \left[ f' - g' + \frac{1}{r} (f+g) \right]^2 \end{aligned}$$

We see that the string  $\hat{T}^{\mu\nu}$  scales as  $1/r$  [eq.(19)] whereas the field  $\hat{T}_\phi^{\mu\nu}$  scales as  $1/r^2$ . Perhaps the slower  $\hat{T}^{\mu\nu}$  decay for strings can be related to the fact that they are extended objects.

#### 4 $\hat{T}^{\mu\nu}$ for massless string states in cylindrically symmetric configurations.

Cosmic strings can be considered as essentially very long straight strings in (almost) cylindrically symmetric configurations. Although they behave as fundamental strings only classically and in the Nambu approximation (that is, zero string thickness), it is important to study the expectation value of  $\hat{T}^{\mu\nu}(x)$  for a cylindrically symmetric configuration. It would be interesting to see if there is an equivalent quantum version of the deficit angle found for cosmic strings [6]. Consider a cylindrically symmetric wave packet

$$\varphi(E, \hat{u}) = \varphi(E, \gamma). \quad (24)$$

We can rewrite the integral (14) in a more convenient way to analyse  $\langle \hat{T}^{\mu\nu}(x) \rangle$  in a cylindrical configuration as follows:

$$\langle \hat{T}^{\mu\nu}(x) \rangle = \frac{1}{24\pi^2} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 \int_0^\pi d\gamma \sin \gamma \int_0^{2\pi} d\alpha \varphi^*(E_1, \gamma) \varphi(E_2, \gamma) e^{i(E_2-E_1)t} e^{i(E_2-E_1)\rho \cos \alpha \sin \gamma} [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + p_1^\mu p_2^\nu] \quad (25)$$

where we have choosen  $\vec{x} = (\rho, 0, 0)$ ,  $\hat{u}_1 \cdot \vec{x} = \rho \cos \alpha \sin \gamma$ , and  $L$  is the radius of a large sphere.

Integrating over  $\alpha$  in eq.(25) we find

$$\begin{aligned} \langle \hat{T}^{00}(t, \rho) \rangle &= \frac{1}{12\pi} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 \int_0^\pi d\gamma \sin \gamma \varphi^*(E_1, \gamma) \varphi(E_2, \gamma) \\ &\quad [E_1^2 + E_2^2 + E_1 E_2] J_0([E_1 - E_2]\rho \sin \gamma) e^{i(E_2-E_1)t}, \\ \langle \hat{T}^{33}(t, \rho) \rangle &= \frac{1}{12\pi} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 \int_0^\pi d\gamma \sin \gamma \cos^2 \gamma \varphi^*(E_1, \gamma) \varphi(E_2, \gamma) \\ &\quad [E_1^2 + E_2^2 + E_1 E_2] J_0([E_1 - E_2]\rho \sin \gamma) e^{i(E_2-E_1)t}, \\ \langle \hat{T}^{03}(t, \rho) \rangle &= \frac{1}{12\pi} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 \int_0^\pi d\gamma \sin \gamma \cos \gamma \varphi^*(E_1, \gamma) \varphi(E_2, \gamma) \\ &\quad [E_1^2 + E_2^2 + E_1 E_2] J_0([E_1 - E_2]\rho \sin \gamma) e^{i(E_2-E_1)t}. \end{aligned} \quad (26)$$

Using rotational invariance around the  $z$ -axis we can express  $\hat{T}^{\alpha\beta}(x)$ ,  $\hat{T}^{3\alpha}(x)$  and  $\hat{T}^{0\alpha}(x)$ ,  $\alpha, \beta = 1, 2$  as follows

$$\begin{aligned} \hat{T}^{\alpha\beta}(x) &= \delta^{\alpha\beta} a(t, \rho) + e^\alpha e^\beta b(t, \rho) \\ \hat{T}^{0\alpha}(x) &= e^\alpha c(t, \rho), \quad \hat{T}^{3\alpha}(x) = e^\alpha d(t, \rho) \end{aligned}$$

where  $e^\alpha = (\cos \phi, \sin \phi)$ . The coefficients  $a(t, \rho)$ ,  $b(t, \rho)$ ,  $c(t, \rho)$  and  $d(t, \rho)$  follow from the calculation for  $\rho = x$  (that is  $\phi = 0$ ):

$$\begin{aligned} a(t, \rho) &= \langle \hat{T}^{22}(t, \rho = x) \rangle, \quad b(t, \rho) = \langle \hat{T}^{11}(t, \rho = x) \rangle - \langle \hat{T}^{22}(t, \rho = x) \rangle, \\ c(t, \rho) &= \langle \hat{T}^{01}(t, \rho = x) \rangle, \quad d(t, \rho) = \langle \hat{T}^{13}(t, \rho = x) \rangle \end{aligned}$$

We find from eqs.(24, 25) at  $\rho = x$ ,

$$\begin{aligned} \langle \hat{T}^{11}(t, \rho = x) \rangle &= \frac{1}{24\pi} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 \int_0^\pi d\gamma \sin^3 \gamma \varphi^*(E_1, \gamma) \varphi(E_2, \gamma) e^{i(E_2-E_1)t} \\ &\quad [E_1^2 + E_2^2 + E_1 E_2] [J_0([E_1 - E_2]\rho \sin \gamma) - J_2([E_1 - E_2]\rho \sin \gamma)], \\ \langle \hat{T}^{22}(t, \rho = x) \rangle &= \frac{1}{12\pi} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 \int_0^\pi d\gamma \sin^2 \gamma \varphi^*(E_1, \gamma) \varphi(E_2, \gamma) \\ &\quad [E_1^2 + E_2^2 + E_1 E_2] J_1([E_1 - E_2]\rho \sin \gamma) \frac{e^{i(E_2-E_1)t}}{(E_1 - E_2)\rho}, \end{aligned}$$

$$\begin{aligned}
\langle \hat{T}^{01}(t, \rho = x) \rangle &= \frac{i}{12\pi} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 \int_0^\pi d\gamma \sin^2 \gamma \varphi^*(E_1, \gamma) \varphi(E_2, \gamma) \\
&\quad \left[ E_1^2 + E_2^2 + E_1 E_2 \right] J_1([E_1 - E_2] \rho \sin \gamma) e^{i(E_2 - E_1)t} , \\
\langle \hat{T}^{13}(t, \rho = x) \rangle &= \frac{i}{12\pi} \int_0^\infty E_1 dE_1 \int_0^\infty E_2 dE_2 \int_0^\pi d\gamma \sin^2 \gamma \cos \gamma \varphi^*(E_1, \gamma) \varphi(E_2, \gamma) \\
&\quad \left[ E_1^2 + E_2^2 + E_1 E_2 \right] J_1([E_1 - E_2] \rho \sin \gamma) e^{i(E_2 - E_1)t} , 
\end{aligned} \tag{27}$$

where  $J_n(z)$  stand for Bessel functions of integer order. Cylindrical symmetry also results in

$$\langle \hat{T}^{02}(t, \rho = x) \rangle = \langle \hat{T}^{12}(t, \rho = x) \rangle = \langle \hat{T}^{23}(t, \rho = x) \rangle = 0.$$

For a wave packet symmetric with respect to the  $xy$  plane,

$$\varphi(E, \gamma) = \varphi(E, \pi - \gamma) ,$$

and we find

$$d(t, \rho) = \langle \hat{T}^{13}(t, \rho = x) \rangle = 0 , \quad \langle \hat{T}^{03}(t, \rho) \rangle = 0 .$$

Notice that

$$2a(t, \rho) + b(t, \rho) + \langle \hat{T}^{33}(t, \rho) \rangle = \langle \hat{T}^{00}(t, r) \rangle$$

due to the tracelessness of the energy-momentum tensor.

In order to compute the asymptotic behaviour  $\rho \rightarrow \infty$ , we change in eq.(26-27) the integration variables

$$E_2 - E_1 = v\tau/\rho \quad , \quad E_2 + E_1 = v .$$

We find for the energy for a real  $\varphi(E, \gamma)$ ,

$$\begin{aligned}
\langle \hat{T}^{00}(t, \rho) \rangle &= \frac{1}{64\pi \rho} \int_0^\infty v^5 dv \int_0^\rho d\tau \int_0^\pi \sin \gamma d\gamma \varphi\left(\frac{v}{2}\left[1 + \frac{\tau}{\rho}\right], \gamma\right) \varphi\left(\frac{v}{2}\left[1 - \frac{\tau}{\rho}\right], \gamma\right) \\
&\quad \cos(v\tau t/\rho) \left[ 1 - \frac{2}{3} \frac{\tau^2}{\rho^2} - \frac{1}{3} \frac{\tau^4}{\rho^4} \right] J_0(v\tau \sin \gamma) .
\end{aligned}$$

This representation is appropriate to compute the limit  $\rho \rightarrow \infty$  with  $t$  fixed. We find in such limit,

$$\langle \hat{T}^{00}(t, \rho) \rangle \xrightarrow{\rho \rightarrow \infty} \frac{1}{2\pi \rho} \int_0^\infty E^4 dE \int_{\sin \gamma > t/\rho} d\gamma \frac{\varphi(E, \gamma)^2}{\sqrt{1 - \left(\frac{t}{\rho \sin \gamma}\right)^2}} ,$$

where we used the formula

$$\int_0^\infty J_0(ax) \cos bx \, dx = \frac{\theta(a^2 - b^2)}{\sqrt{a^2 - b^2}} .$$

We can further simplify  $\langle \hat{T}^{00}(t, \rho) \rangle$  for  $\rho \gg t$  yielding

$$\langle \hat{T}^{00}(t, \rho) \rangle \xrightarrow{t \rightarrow \infty} \frac{1}{2\pi \rho} \int_0^\infty E^4 dE \int_0^\pi d\gamma \varphi(E, \gamma)^2 + O\left(\frac{t^2}{\rho^2}\right) .$$

That is, the energy decays as  $\rho^{-1}$  for large  $\rho$  and fixed  $t$ .

We analogously compute the  $\rho \rightarrow \infty$ ,  $t$  fixed behaviour of the other components of  $\langle \hat{T}^{\mu\nu}(x) \rangle$  with the following results

$$\begin{aligned}
a(t, \rho) &\stackrel{\rho \rightarrow \infty}{=} \frac{1}{2\pi \rho} \int_0^\infty E^4 dE \int_{\sin \gamma > t/\rho} d\gamma \varphi(E, \gamma)^2 \sin^2 \gamma \sqrt{1 - \left(\frac{t}{\rho \sin \gamma}\right)^2}, \\
a(t, \rho) + b(t, \rho) &\stackrel{\rho \rightarrow \infty}{=} \frac{t^2}{2\pi \rho^3} \int_0^\infty E^4 dE \int_{\sin \gamma > t/\rho} d\gamma \frac{\varphi(E, \gamma)^2}{\sqrt{1 - \left(\frac{t}{\rho \sin \gamma}\right)^2}} \\
&\stackrel{\rho \rightarrow \infty}{=} \frac{t^2}{\rho^2} \langle \hat{T}^{00}(t, \rho) \rangle, \\
\langle \hat{T}^{33}(t, \rho) \rangle &\stackrel{\rho \rightarrow \infty}{=} \frac{1}{2\pi \rho} \int_0^\infty E^4 dE \int_{\sin \gamma > t/\rho} d\gamma \cos^2 \gamma \frac{\varphi(E, \gamma)^2}{\sqrt{1 - \left(\frac{t}{\rho \sin \gamma}\right)^2}}, \\
c(t, \rho) &\stackrel{\rho \rightarrow \infty}{=} \frac{t}{2\pi \rho^2} \int_0^\infty E^4 dE \int_{\sin \gamma > t/\rho} d\gamma \frac{\varphi(E, \gamma)^2}{\sqrt{1 - \left(\frac{t}{\rho \sin \gamma}\right)^2}} \\
&\stackrel{\rho \rightarrow \infty}{=} \frac{t}{\rho} \langle \hat{T}^{00}(t, \rho) \rangle.
\end{aligned}$$

The calculation of the  $t \rightarrow \infty$  behaviour for  $\rho$  fixed can be easily obtained from eq.(26) undoing the integration over  $\alpha$ . That is,

$$\langle \hat{T}^{00}(t, \rho) \rangle = \frac{1}{24\pi^2} \int_0^\pi d\gamma \sin \gamma \int_0^{2\pi} d\alpha [f_3(\gamma, \alpha) f_1^*(\gamma, \alpha) + f_1(\gamma, \alpha) f_3^*(\gamma, \alpha) + f_2(\gamma, \alpha) f_2^*(\gamma, \alpha)] ; \quad (28)$$

where

$$f_n(\gamma, \alpha) \equiv \int_0^\infty E^n dE \varphi(E, \gamma) e^{iE(\rho \sin \gamma \sin \alpha - t)}. \quad (29)$$

We explicitly see here  $\langle \hat{T}^{00}(t, \rho) \rangle$  as a superposition of outgoing and ingoing cylindrical waves.

The integral (29) is dominated in the  $t \rightarrow \infty$  limit by its lower bound. We find,

$$f_n(\gamma, \alpha) \stackrel{t \rightarrow \infty}{=} n! (-i)^{n+1} \varphi(0, \gamma) t^{-n-1} [1 + O(t^{-1})].$$

Inserting this result in eq.(28) yields

$$\langle \hat{T}^{00}(t, \rho) \rangle \stackrel{t \rightarrow \infty}{=} -\frac{2}{3\pi t^6} \int_0^\pi d\gamma \sin \gamma \varphi(0, \gamma)^2 [1 + O(t^{-1})].$$

Using the same technique we find for the  $t \rightarrow \infty$  behaviour for  $\rho$  fixed of the other  $\langle \hat{T}^{\mu\nu}(t, \rho) \rangle$  components,

$$\begin{aligned}
\langle \hat{T}^{33}(t, \rho) \rangle &\stackrel{t \rightarrow \infty}{=} -\frac{2}{3\pi t^6} \int_0^\pi d\gamma \sin \gamma \cos^2 \gamma \varphi(0, \gamma)^2 [1 + O(t^{-1})], \\
a(t, \rho) &\stackrel{t \rightarrow \infty}{=} -\frac{1}{3\pi t^6} \int_0^\pi d\gamma \sin^3 \gamma \varphi(0, \gamma)^2 [1 + O(t^{-1})],
\end{aligned}$$

$$\begin{aligned} b(t, \rho) &\stackrel{t \rightarrow \infty}{=} 0 + O(t^{-7}) , \\ c(t, \rho) &\stackrel{t \rightarrow \infty}{=} 0 + O(t^{-7}) . \end{aligned}$$

As before, a typical form of the wave function  $\varphi(E, \gamma)$  is a gaussian Ansatz

$$\varphi(E, \gamma) = \left(\frac{2c}{\pi}\right)^{3/2} e^{-E^2(a^2 \sin^2 \gamma + b^2 \cos^2 \gamma)} \quad (30)$$

where  $a^2$  and  $b^2$  are constants.

From the expressions above we can see that for large  $t$  and  $\rho$  fixed the energy density decays rapidly as  $O(1/t^6)$  whilst the energy flux vanishes. For  $\rho$  large and fixed  $t$  the situation is similar to the spherical one, the energy density decays as  $1/\rho$  and the energy flux decays as  $1/\rho^2$ .

If we go now to regions where both  $\rho$  and  $t$  are small, we find after a careful examination of  $\langle \hat{T}^{\mu\nu} \rangle$  that **all** of its components are finite constants in such limit.

## 5 Final remarks.

An important point we need to stress, is the fact that, in contrast to what happens in the classical theory where there is a logarithmic divergence as we approach the core of the string [3], in the quantum theory we can obtain a metric where, if any divergences are present these are not related at all to the position of the string [10]. We showed, that the localised behaviour of the string given by  $\delta(x - X(\sigma, \tau))$  disappears when we take into account the quantum nature of the strings. In the *classical theory of radiating strings*, divergences coincide with the source of the gravitational field. However, as we have seen, quantum mechanically this is not the case. The source position is smeared by the quantum fluctuations. The probability amplitude for the centre of mass string position is given by the Fourier transform of  $\varphi(\vec{p})$  and is a smooth function peaked at the origin.

We show that **all** components of  $\langle \hat{T}^{\mu\nu} \rangle$  are finite everywhere both in the cylindrically and in the spherically symmetric cases. [This finiteness was already observed in ref.[4] for shock-wave spacetimes]. The present results support the belief that string theory is a finite theory.

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